Last time: More basic logic.
Logical equivalence.

Addendum: Add to your table of Equivalences:

Idempotence: $P \wedge P \equiv P$, $P \vee P \equiv P$

Remark: There is some redundancy in our table of equivalences...

Ex: Is $\neg(P \to Q)$ logically equivalent to $(\neg P) \to (\neg Q)$?

Sol:

P	Q	$\neg P$	$\neg Q$	$P \to Q$	$\neg(P \to Q)$	$(\neg P) \to (\neg Q)$
T	T	F	F	T	F	T
T	F	F	T	F	T	T
F	T	T	F	T	F	F
F	F	T	T	T	F	T

Therefore $\neg(P \to Q) \not\equiv (\neg P) \to (\neg Q)$ by definition ∎

Algebra of Logical Equivalence

IDEA: Build from the table of equivalences to find new equivalences.

Ex: We use the basic equivalences to derive

$$\neg(P \to Q) \equiv P \wedge (\neg Q).$$

In particular

$$\neg(P \to Q) \equiv \neg((\neg P) \vee Q) \quad \text{(Material implication)}$$

Ex: We show $\neg(P \to Q) \equiv P \wedge (\neg Q)$ via string of equivs.

$$\neg(P \to Q)$$
$$\equiv \neg((\neg P) \vee Q) \quad \text{(material implication)}$$
$$\equiv (\neg(\neg P)) \wedge (\neg Q) \quad \text{(De Morgan's Law)}$$
$$\equiv P \wedge (\neg Q) \quad \text{(Double Negation)}$$

Ex: Show $(P \to Q) \vee (P \to R) \equiv P \to (Q \vee R)$ using the algebra of equivalences.

Sol:
$$(P \to Q) \vee (P \to R)$$
$$\equiv ((\neg P) \vee Q) \vee (P \to R) \quad \text{(Material Implication)}$$
$$\equiv ((\neg P) \vee Q) \vee ((\neg P) \vee R) \quad \text{(Material Implication)}$$
$$\equiv (\neg P) \vee (Q \vee ((\neg P) \vee R)) \quad \text{(Associativity)}$$
$$\equiv (\neg P) \vee ((Q \vee (\neg P)) \vee R) \quad \text{(Assoc.)}$$
$$\equiv (\neg P) \vee ((\neg P) \vee Q) \vee R) \quad \text{(commutativity)}$$
$$\equiv (\neg P) \vee ((\neg P) \vee (Q \vee R)) \quad \text{(Assoc)}$$
$$\equiv ((\neg P) \vee (\neg P)) \vee (Q \vee R) \quad \text{(Assoc)}$$
$$\equiv (\neg P) \vee (Q \vee R) \quad \text{(Idempotence)}$$
$$\equiv P \to (Q \vee R) \quad \text{(Mat. Imp.)}$$

Crucial Observation: This is only good for proving equivalence...

Soon: we'll learn how to prove implications...

NB: Proving non-equivalence using statement algebra requires one more piece of knowledge: Disjunctive Normal Forms...

Ex: Build the disjunctive normal form of $P \to Q$ (from its truth table):

P	Q	$P \to Q$	
T	T	T	$P \wedge Q$ ← true only in 1st row
T	F	F	← False, so ignore...
F	T	T	$(\neg P) \wedge Q$ ← true only in 3rd row
F	F	T	$(\neg P) \wedge (\neg Q)$ true only in 4th row

So the disjunctive normal form is the OR- of these statements

i.e. $$P \to Q \equiv (P \wedge Q) \vee ((\neg P) \wedge Q) \vee ((\neg P) \wedge (\neg Q)).$$

Exercise: Verify this equivalence via truth table. V.

Prop: There is a unique disjunctive normal form for any given statement in the propositional logic.

The DNF is constructed (from the truth table) by

① Identifying all "true rows" of the table.

② Constructing the conjunction associated to each "true row"

③ Taking the "total or" of these "true row conjunctions"

Prop: Our logical equivalence table is "logically complete".

I.E. Any two equivalent (propositional) statements can be connected via a string of basic equivalences.

NB: Thus we can compare DNF's (= "disjunctive normal form") computed via basic equivalences to decide statements are (non-) equivalent.

Arguments and Argument Forms

An **argument** is a sequence of (propositional) deductions.

An argument is ...

- **valid** when assigning true to all premises yields a true conclusion.
- **invalid** otherwise.

A **premise** is an assumed/given statement. The **conclusion** of an argument is its final deduced statement.

Ex: [If the dragon eats meat, then the dragon is not vegan.] [The dragon does eat meat.] [Thus far, the dragon is not vegan.]

premises. conclusion.

In the propositional world, this translates to:

({ M : the dragon eats meat
  { V : the dragon is vegan. ← dictionary)

$[M \to (\neg V),\ M,\ \therefore \neg V]$ ← argument form. ☑

Notation: If an argument form has premises $P_1, P_2, \dots, P_n$ and conclusion q, we denote the form by

$$P_1, P_2, \dots, P_n \therefore q$$

Test for validity: Argument form $P_1, P_2, \dots, P_n \therefore q$ is valid precisely when $(P_1 \wedge P_2 \wedge \dots \wedge P_n) \longrightarrow q$ is a tautology.

Ex: Analyzing the previous argument $(M \to (\neg V),\ M \therefore \neg V)$

M	V	¬V	M → (¬V)	(M → (¬V)) ∧ M	((M → (¬V)) ∧ M) → (¬V)
T	T	F	F	F	T
T	F	T	T	T	T
F	T	F	T	F	T
F	F	T	T	F	T

∴ this argument form is valid by our validity test.

Q: Why $((M \to (\neg V)) \wedge M) \to \neg V$ in the last column?

$M \to (\neg V)$, M $\therefore \neg V$ is <u>valid</u> precisely when
P_1 P_2 q

$(P_1 \wedge P_2) \to q$ is tautological

Basic Valid Argument Forms

Name	Argument Form
Modus Ponens	$P \to Q$, P $\therefore Q$
Modus Tollens	$P \to Q$, $\neg Q$ $\therefore \neg P$
Disjunctive Syllogism	$P \vee Q$, $\neg P$ $\therefore Q$
Hypothetical Syllogism	$P \to Q$, $Q \to R$ $\therefore P \to R$
Dilemma	$P \vee Q$, $P \to R$, $Q \to R$ $\therefore R$
Reductio Ad Absurdum	$(\neg P) \to (Q \wedge (\neg Q))$ $\therefore P$
Conjunctive Simplification	$P \wedge Q$ $\therefore P$
Conjunctive Addition	P, Q $\therefore P \wedge Q$
Disjunctive Addition	P $\therefore P \vee Q$
Biconditional Syllogism	$P \leftrightarrow Q$, $Q \leftrightarrow R$ $\therefore P \leftrightarrow R$

<u>Exercise</u>: Verify the validity of each argument above.